Allocation Processes for Worst Scenarios in Fuzzy Asset Management Using Weighted Average Value-at-Risks

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***Abstract*— A dynamic portfolio allocation is discussed in asset management with fuzziness. By perception-based extension for fuzzy random variables, a dynamic portfolio model for weighted average value-at-risks of fuzzy random variables is introduced. By mathematical programming and dynamic programming, this paper derives analytical solutions of the optimization problem for dynamic worst scenarios. A few numerical examples are given to discuss the obtained results. It is shown that the weighted average value-at-risk gives a more reasonable criterion than value-at-risk.**

***Index Terms*— Weighted average value-at-risk, risk-sensitive portfolio, fuzzy random variable, perception-based extension, portfolio allocation, drastic decline, pessimistic-optimistic index, possibility-necessity weight.**

I. Introduction

THIS paper deals with a dynamic allocation model with portfolio optimization in fuzzy asset management. When the financial crisis in September 2008 and the China's stock market crashing in May 2015, we have experienced the serious distrust about the stock market because of doubtful information among investors and the market. No one can foretell that similar financial crisis will never happen in future. Some of the serious distrust observed at that time and the risky information regarding banks and security companies are coming from the imprecision of information. The fuzzy logic is a tool to describe the imprecision of data because of a lack of knowledge, and such serious distrust in the stock market will be represented by the fuzziness of information in finance models. Optimization in fuzzy logic framework was studied firstly as decision-making with fuzzy goal, which was introduced by Bellman and Zadeh [25], and the portfolio optimization with fuzziness has been developed with possibility measures and necessity measures by Tanaka and Guo [15], Tanaka et al. [16], Watada [18], Katagiri at al. [4] and so on. Soft computing like fuzzy logic works effectively for finance models in uncertain environments. To represent the uncertainty, in this paper we use fuzzy random variables which have two kinds of uncertainties, i.e. randomness and fuzziness. In this model, randomness is used to represent the uncertainty regarding the belief degree of frequency, and fuzziness is applied to description regarding linguistic imprecision of data because of a lack of knowledge about the current stock market. Fuzzy random variables, which were introduced by Kwakernaak [7], are applied to decision-making under uncertainty with fuzziness like linguistic data in statistics, engineering and economics. By using the perception-based approach in Yoshida [20],[21], the criteria for real-valued random variables are extended to a criteria for fuzzy random variables, and this paper applies perception-based criteria to a dynamic optimization model with uncertainty. Yoshida [19] studied the defuzzification of the mean and the variance of fuzzy random variables, and then fuzzy random variables are evaluated by the expectation and these criteria with pessimistic-optimistic parameters and possibility-necessity parameters. These parameters are given by the decision maker and they are based on his certainty about information in the stock market

Portfolio theory in financial engineering is a powerful technique to hedge risks in asset management, and the mean and variance portfolio model, where the expected rewards are maximized and the risks are minimized, is studied by many researchers ([8],[9],[12],[15]). Then the variance was used as a criterion of risks in classic models. Recently drastic declines of asset prices are studied, and value-at-risk (VaR) is used widely to estimate risks when asset prices decline rapidly with worst scenarios. Nowadays value-at-risk is a standard tool to indicate the risk concerning worst scenarios in financial investment. However it does not have coherency. Coherent risk measures have been studied to improve the criterion of risks with worst scenarios by Artzner et al. [1] and so on. Several improved risk measures based on value-at-risks are proposed: for example, conditional value-at-risk, expected shortfall, entropic value-at-risk (Rockafellar and Uryasev [11]; Tasche [13]). Kusuoka [6] also gave a spectral representation for coherent risk measures. Average value-at-risk defined by value-at-risks is a coherent risk measure, and Yoshida [24] have discussed a portfolio optimization problem for dynamic worst scenarios with average value-at-risks of the returns with fuzziness using the results in Yoshida [19],[21],[22],[23].

This paper focuses on decision making to obviate the worst sharp decline such as worldwide stock market flash crash which have heavy impact for a long time. Using weighted average

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value-at-risks and dynamic programming, we obtain analytical solutions for risk-minimization of dynamic systems under uncertainty, and further we investigate concrete solutions in case of normal distributions. In numerical example, we compare three criteria given by value-at-risks, average value-at-risks and weighted average value-at-risks.

In Section 2, we introduce a portfolio allocation model and we discuss the drastic decline of asset prices using value-at-risks. In Section 3, we introduce fuzzy numbers and fuzzy random variables, and we discuss weighted average value-at-risks and their extension for fuzzy random variables and dynamic systems. In Section 4, we give estimation tools with lambda-mean functions and evaluation weights in order to evaluate the randomness and fuzziness of fuzzy random variables. In Section 5, we introduce a dynamic portfolio allocation model using fuzzy random variables. We investigate risk-minimizing problems with weighted average value-at-risks at a current time by mathematical programming, and we discuss the dynamic portfolio optimization by dynamic programming. In Section 6 we obtain concrete solutions in case of normal distributions. In Section 7 we give a few numerical examples and we investigate the solutions of the dynamic portfolio optimization problem. Then we find the weighted average value-at-risk gives a more reasonable criterion than value-at-risk and average value-at-risk.

II. Decision-making with portfolio weights

We introduce dynamic portfolio decision-making with n risky securities and an expiration date T, where n and T are positive integers. Let P be a non-atomic probability measure on a sample space Ω, and let M be a sigma-field of Ω. For an asset i = 1, 2, $\cdots$, n, a dynamic stock price is given by a sequence of real-valued random variables $\{ S_t^i \}$, where the initial stock price S_0^i is a positive number, and the rates of return R_t^i are defined by real-valued random variables satisfying

$$S_t^i = S_{t-1}^i (1 + R_t^i), \tag{1}$$

and then $1+R_t^i$ is non-negative. Let $\mathbf{R}$ be the set of all real numbers. We deal with portfolio weight vectors $(w_t^1, w_t^2, ..., w_t^n)$ in $\mathbf{R}^n$ satisfying $\sum_i w_t^i = 1$ and w_t^i are non-negative for all asset i. At time t, the rate of return with a portfolio weight vector $(w_t^1, w_t^2, ..., w_t^n)$ is given by $R_t = \sum_i w_t^i R_t^i$ and then we give

$$S_t = S_{t-1}(1 + R_t) \tag{2}$$

for each time t. We discuss the sharp decline of prices in asset management ([23]). A scenario ω in Ω such that the theoretical bankruptcy happens at time t implies $S_{t-1}(\omega)$ is positive and $S_t(\omega)$ is non-positive, and then we have $R_t(\omega) \le -1$ from (2). In a similar idea, for a constant δ between 0 and 1,

$$p = P(R_t \le -\delta) \tag{3}$$

is a probability of 100δ %-falling. On the other hand for a positive probability p, a value-at-risk is a real number v satisfying

$$p = P(R_t \le v) . \tag{4}$$

From (3) and (4), we get $-v = \delta$. Thus throughout this paper we note that the minimization of risk values δ is equivalent to maximization of value-at-risks v.

III. Fuzzy random variables and the criteria

In this paper, we represent fuzzy numbers on $\mathbf{R}$ by membership functions $\tilde{a}$, which is a mapping from $\mathbf{R}$ to [0,1], and their α-cuts are represented by closed intervals $\tilde{a}_\alpha$ = $[\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ ([25]). For fuzzy numbers $\tilde{a}$ and $\tilde{b}$, fuzzy max order $\tilde{a} \ge \tilde{b}$ implies $\tilde{a}_\alpha^\pm \ge \tilde{b}_\alpha^\pm$ for all α between 0 and 1 ([19]). For a positive probability p and an integrable real-valued random variable X on Ω with a strictly increasing and continuous cumulative distribution function F_X , the value-at-risk is defined by a percentile $\mathrm{VaR}_p = F_X^{-1}(p)$, which is given by the inverse function of F_X , and then the average value-at-risk and the weighted average value-at-risk are defined respectively by

$$\mathrm{AVaR}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\ dq, \tag{5}$$

$$\mathrm{WAVaR}_p(X) = \int_0^p \mathrm{VaR}_q(X)\, h(q)\, dq \Big/ \int_0^p h(q)\, dq, \tag{6}$$

where h is a non-increasing and nonnegative-valued weight function on (0,1] whose integrand equals to 1. It is known in Artzner et al. [1] that the risk measure $-\mathrm{VaR}p$ is not coherent however $-\mathrm{AVaR}p$ and $-\mathrm{WAVaR}p$ are coherent risk measures. This paper estimates the rates of return by coherent risk measures $-\mathrm{AVaR}p$ and $-\mathrm{WAVaR}p$.

A map $\tilde{X}$ from Ω to the set of all fuzzy numbers is called a fuzzy random variable if $\tilde{X}_\alpha^\pm$ are real-valued random variables for all α, where the α-cuts of $\tilde{X}$ are $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)]$ for ω. A perception-based expectation of a fuzzy random variable is a fuzzy number given by

$$\tilde{E}(\tilde{X}) = \sup_{X:E(X)=x} \inf_{\omega} \tilde{X}(\omega)(X(\omega)) \tag{7}$$

for x in $\mathbf{R}$, where $E(\ )$ is the expectation of real-valued random variables ([5]). Similarly, for a positive probability p, average value-at-risk and weighted average value-at-risk of fuzzy random variables are introduced respectively by

$$\mathrm{AVaR}_p(\tilde{X})(x) = \sup_{X:\mathrm{AVaR}_p(X)=x} \inf_{\omega} \tilde{X}(\omega)(X(\omega)) , \tag{8}$$

$$\mathrm{WAVaR}_p(\tilde{X})(x) = \sup_{X:\mathrm{WAVaR}_p(X)=x} \inf_{\omega} \tilde{X}(\omega)(X(\omega)) \tag{9}$$

for x in $\mathbf{R}$. These extended criteria (8) and (9) have the following properties ([21]).

Lemma 1. For a positive probability p, AVaR_p and WAVaR_p are monotone, positively homogeneous and translation invariant.

Let $\mathcal{G}$ be a sub-sigma-field of $\mathcal{M}$ and let $E(\ \mid \mathcal{G})$ be the conditional expectation. We give value-at-risk and weighted average value-at-risk of real-valued random variables X under condition $\mathcal{G}$ as follows:

$$\mathrm{VaR}_p(\ \mid \mathcal{G}) = \sup\{\, x \in \mathbb{R} \mid E(1_{\{X<x\}} \mid \mathcal{G}) \le p \} \tag{10}$$

with the characteristic function $1_{\{\ \}}$, and

$$\mathrm{WAVaR}_p(X \mid \mathcal{G}) = \int_0^p \mathrm{VaR}_q(X \mid \mathcal{G}) h(q)\,dq \Big/ \int_0^p h(q)\,dq. \tag{11}$$

Hence we can give $\mathcal{G}$-measurable fuzzy random variables $\mathrm{WAVaR}_p(\ \mid \mathcal{G})$, which is defined in the same way as (9), and then $\mathrm{WAVaR}_p(\ \mid \mathcal{G})$ has the following similar properties.

Lemma 2. Let p be a positive probability and let $\tilde{X}$ and $\tilde{Y}$ be fuzzy random variables. Let $\mathcal{G}$ be a sigma-field generated by $\tilde{Y}$. Assume $\tilde{X}$ and $\mathcal{G}$ are independent. Then $\mathrm{WAVaR}_p(\ \mid \mathcal{G})$ is monotone, positively homogeneous and translation invariant, and the following (i) and (ii) hold.

(i) $\mathrm{WAVaR}_p(\tilde{X} \mid \mathcal{G}) = \mathrm{WAVaR}_p(\tilde{X})$.

(ii) $\mathrm{WAVaR}_p(\tilde{Y} \mid \mathcal{G}) = \tilde{Y}$.

IV. Estimations of Fuzziness

Defuzzification of fuzzy numbers is studied by many authors ([3]). Here we introduce a defuzzification method in [19]. Let

$$g^{\lambda}([x, y]) = \lambda \cdot x + (1-\lambda) \cdot y \tag{12}$$

for closed intervals $[x, y]$, where λ is a constant between 0 and 1. Then (12) is called a lambda-mean function and λ is called the pessimistic index if λ is 1, and it is called the optimistic index if λ is 0 ([3]). We give a mean value of a fuzzy number $\tilde{a}$ by lambda-mean function (12) and an evaluation weight $w(\alpha)$ as follows ([19]):

$$E^{\lambda}(\tilde{a}) = \int_0^1 g^{\lambda}(\tilde{a}_{\alpha})\, w(\alpha)\, d\alpha \Big/ \int_0^1 w(\alpha)\, d\alpha. \tag{13}$$

Hence, $w(\alpha)$ is called the possibility evaluation weight if $w(\alpha)$ is 1 for α between 0 and 1, and it is called the necessity evaluation weight if $w(\alpha)$ is $1-\alpha$ for α between 0 and 1. For λ between 0 and 1, the mean E^{λ} has the following natural properties ([19]).

Lemma 3. ([19]). The expectation $E^{\lambda}(\ )$ is monotone, positively homogeneous, additive and translation invariant.

For a fuzzy random variable $\tilde{X}$, the expectation of the mean $E(E^{\lambda}(\tilde{X}))$ follows

$$E(E^{\lambda}(\tilde{X})) = E\left(\int_0^1 g^{\lambda}(\tilde{X}_{\alpha})\, w(\alpha)\, d\alpha \Big/ \int_0^1 w(\alpha)\, d\alpha \right). \tag{14}$$

From Lemma 3, we obtain the following results for (14) ([19]).

Lemma 4. Let $\tilde{X}$ be a fuzzy random variable, let Z be a real-valued random variable and let $\tilde{a}$ be a fuzzy number. Then $E(E^{\lambda}(\ ))$ is monotone, positively homogeneous and additive, and the following (i) - (iii) hold.

(i) $E(E^{\lambda}(\tilde{X})) = E^{\lambda}(E(\tilde{X}))$.

(ii) $E(E^{\lambda}(\tilde{a})) = E^{\lambda}(\tilde{a})$.

(iii) $E(E^{\lambda}(Z)) = E(Z)$.

Let $\mathcal{A}$ be a family of fuzzy random variables $\tilde{X}$ for which there exist a real-valued random variable X and a fuzzy number $\tilde{a}$ such that $\tilde{X}_{\alpha}(\omega) = X(\omega) + \tilde{a}_{\alpha}$ for ω and α between 0 and 1. Lemma 4(i) implies the expectation $E(\ )$ and the mean $E^{\lambda}(\ )$ are exchangeable. On the other hand, the following proposition can be easily checked by (6), (9) and Lemma 1.

Proposition 1. For a positive probability p and a fuzzy random variable $\tilde{X}$ in $\mathcal{A}$, it holds that

$$E^{\lambda}(\mathrm{WAVaR}_p(\tilde{X})) = \mathrm{WAVaR}_p(E^{\lambda}(\tilde{X})). \tag{15}$$

V. Dynamic Weighted Average Value-at-Risks and Fuzzy Random Variables

We introduce a dynamic portfolio allocation model with fuzzy random variables. Let R_t^i be the rates of return in Section 2. We deal with the following fuzzy random variable in $\mathcal{A}$:

$$(\tilde{R}_t^i(\omega))_{\alpha} = R_t^i(\omega) + (\tilde{a}_t^i)_{\alpha} \tag{16}$$

for ω in Ω and α between 0 and 1, where $\tilde{a}_t^i$ is a fuzzy number. Hence we assume

$$1 + \tilde{R}_t^i \ge 0 \tag{17}$$

for all asset i and time t. For a portfolio $(w_t^1, w_t^2, \ldots, w_t^n)$, the rate of return with the portfolio follows

$$\tilde{R}_t = \sum_i w_t^i \tilde{R}_t^i, \tag{18}$$

and we give the asset price by $\tilde{S}_t = \tilde{S}_{t-1}(1 + \tilde{R}_t)$ for t = 1, 2, …, T. We assume the initial asset price is a real number $\tilde{S}_0 = 1$ for simplicity. Then we have

$$\tilde{S}_t = \prod_{s:s \le t} (1 + \tilde{R}_s) \tag{19}$$

for t = 1, 2, …, T. Let $\mathcal{M}_t$ be the sigma-field generated by random variables $\{\, \tilde{R}_s^i \mid s = 1, 2, \ldots, t;\ i = 1, 2, \ldots, n \,\}$ for positive t and let $\mathcal{M}_0 = \{\varnothing, \Omega\}$. Hence we assume random variables R_t^i is independent of the past information $\mathcal{M}_{t-1}$. Then we have $\mathrm{WAVaR}_p(\tilde{S}_t \mid \mathcal{M}_{t-1}) = \tilde{S}_{t-1}(1 + \mathrm{WAVaR}_p(\tilde{R}_t))$ from Lemmas 1 and 2. Hence we deal with the following portfolio optimization problem for worst scenarios with weighted average value-at-risks. Let λ between 0 and 1, and let a positive discount rate β.

Problem 1. Maximize the dynamic minimum regarding weighted average value-at-risks of the asset prices:

$$\min_{t:1\le t\le T} \beta^{t-1} \prod_{s:s\le t-1} (1+E(E^{\lambda}(\tilde{R}_s)))(1+\mathrm{WAVaR}_p(E^{\lambda}(\tilde{R}_t))) \quad (20)$$

with portfolios $(w_t^1, w_t^2, ..., w_t^n)$.

VI. Dynamic portfolio optimization with weighted average value-at-risks

Firstly we investigate the rates of return with portfolios at time t. Let the mean and the covariance of the rates of return $\tilde{R}_t^i$ by $\mu_t^i = E(E^{\lambda}(\tilde{R}_t^i))$ and $\sigma_t^{ij} = Cov(E^{\lambda}(\tilde{R}_t^i), E^{\lambda}(\tilde{R}_t^j))$ for i, j =1, 2, … , n. Let

$$\mu_t = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix},\ \Sigma_t = \begin{bmatrix} \sigma_t^{11} & \sigma_t^{12} & \cdots & \sigma_t^{1n} \\ \sigma_t^{21} & \sigma_t^{22} & \cdots & \sigma_t^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \sigma_t^{n2} & \cdots & \sigma_t^{nn} \end{bmatrix},\ \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$A_t = \mathbf{1}^{\mathrm{T}} \Sigma_t^{-1} \mathbf{1}$, $B_t = \mathbf{1}^{\mathrm{T}} \Sigma_t^{-1} \mu_t$, $C_t = \mu_t^{\mathrm{T}} \Sigma_t^{-1} \mu_t$ and $\Delta_t = A_t C_t - B_t^2$, where T implies the transpose of a vector and we assume the determinant of the matrix Σ_t is not zero. Then there exists an inverse matrix Σ_t^{-1} and A_t is positive. This assumption is natural and it can be realized easily by taking care of the combination of assets. For a portfolio $(w_t^1, w_t^2, ..., w_t^n)$, the expectation and the variance are given respectively by

$$E(E^{\lambda}(\tilde{R}_t)) = \sum_i w_t^i \mu_t^i \text{ and } \sum_{i,j} w_t^i w_t^j \sigma_t^{ij}. \quad (21)$$

This paper discusses portfolio optimization where the value-at-risk $\mathrm{VaR}_p(E^{\lambda}(\tilde{R}_t))$ has the following representation:

$$\mathrm{VaR}_p(E^{\lambda}(\tilde{R}_t)) = \text{(the mean)} - \kappa(p) \times \text{(the standard deviation)}, \quad (22)$$

where the positive constant $\kappa(p)$ is given corresponding to probability p. One of the sufficient conditions for (22) is what the rates of return $\tilde{R}_t^i$ have normal distributions. From (6),(21), (22) and Proposition 1 we have

$$\mathrm{WAVaR}_p(E^{\lambda}(\tilde{R}_t)) = \sum_i w_t^i \mu_t^i - \kappa \sqrt{\sum_{i,j} w_t^i w_t^j \sigma_t^{ij}}, \quad (23)$$

where

$$\kappa = \int_0^p \kappa(q)\, h(q)\, dq \Big/ \int_0^p h(q)\, dq. \quad (24)$$

Then by dynamic programming we obtain the following theorem.

Theorem 1. Let a sequence $\{ v_t \}$ by the following optimality equations:

$$v_t = \max_{(w_t^1,...,w_t^n)} \min \begin{Bmatrix} 1 + \sum_i w_t^i \mu_t^i - \kappa \sqrt{\sum_{i,j} w_t^i w_t^j \sigma_t^{ij}}, \\ (1 + \sum_i w_t^i \mu_t^i) \beta v_{t+1} \end{Bmatrix} \quad (25)$$

for $t = 1, 2, …, T-1$, and

$$v_T = \max_{(w_T^1,...,w_T^n)} \left(1 + \sum_i w_T^i \mu_T^i - \kappa \sqrt{\sum_{i,j} w_T^i w_T^j \sigma_T^{ij}} \right). \quad (26)$$

Then v_1 is the optimal weighted average value-at-risk for Problem 1.

Hence we assume Δ_t is positive and κ satisfies $\kappa^2 > \Delta_t / A_t$ for all t. Now we deal with the following optimization at a fixed time t to get concrete representation for dynamic equation (25).

Problem 2. Maximize weighted average value-at-risk

$$\mathrm{WAVaR}_p(E^{\lambda}(\tilde{R}_t)) = \sum_i w_t^i \mu_t^i - \kappa \sqrt{\sum_{i,j} w_t^i w_t^j \sigma_t^{ij}} \quad (27)$$

with potofolios $(w_t^1, w_t^2, ..., w_t^n)$.

In a similar way to [24], we get the following lemma for Problem 2.

Lemma 5. The maximum weighted average value-at-risk for Problem 2 is

$$\frac{B_t - \sqrt{A_t \kappa^2 - \Delta_t}}{A_t}. \quad (28)$$

Then the corresponding expected rates of return is

$$\gamma_t^* = \frac{B_t}{A_t} + \frac{\Delta_t}{A_t \sqrt{A_t \kappa^2 - \Delta_t}}, \quad (29)$$

and the optimal portfolio is given by

$$w_t^* = \xi_t^* \Sigma_t^{-1} \mathbf{1} + \eta_t^* \Sigma_t^{-1} \mu_t \quad (30)$$

with

$$\xi_t^* = \frac{C_t - B_t \gamma_t^*}{\Delta_t} \text{ and } \eta_t^* = \frac{A_t \gamma_t^* - B_t}{\Delta_t}. \quad (31)$$

Now we go back to the dynamic optimization problem. Finally we can check the following theorem from Theorem 1 and Lemma 5.

Theorem 2. Assume A_t, B_t and Δ_t are positive, and let κ satisfy $\kappa^2 > \Delta_t / A_t$ for all t. Let $\{ \gamma_t^* \}$ and $\{ v_t \}$ be sequences defined inductively by the following backward optimality equations:

$$\gamma_t^* = \begin{cases} \dfrac{B_t}{A_t} + \dfrac{\Delta_t}{A_t\sqrt{A_t\kappa^2 - \Delta_t}} \\ \qquad \text{if } A_t + 2B_t + C_t \le \dfrac{\kappa^2}{(1-\beta v_{t+1})^2} \\ \max\limits_{l=1,2} \dfrac{B_t + \Gamma_t + (-1)^l \sqrt{(A_t + 2B_t + C_t)\Gamma_t - \Delta_t}}{A_t - \Gamma_t} \\ \qquad \text{if } A_t + 2B_t + C_t > \dfrac{\kappa^2}{(1-\beta v_{t+1})^2} \text{ and } A_t \ne \Gamma_t \\ \dfrac{C_t - A_t}{2(A_t + B_t)} \\ \qquad \text{if } A_t + 2B_t + C_t > \dfrac{\kappa^2}{(1-\beta v_{t+1})^2} \text{ and } A_t = \Gamma_t, \end{cases} \tag{32}$$

$$v_t = \begin{cases} \dfrac{A_t + B_t - \sqrt{A_t\kappa^2 - \Delta_t}}{A_t} \\ \qquad \text{if } A_t + 2B_t + C_t \le \dfrac{\kappa^2}{(1-\beta v_{t+1})^2} \\ (1+\gamma_t^*)\beta\, v_{t+1} \qquad \text{otherwise} \end{cases} \tag{33}$$

for $t = 1, 2, \ldots, T-1$, and

$$v_T = 1 + \frac{B_T - \sqrt{A_T\kappa^2 - \Delta_T}}{A_T}, \tag{34}$$

where

$$\Gamma_t = \frac{\Delta_t\,(1-\beta v_{t+1})^2}{\kappa^2}. \tag{35}$$

Then v_1 is the optimal weighted average value-at-risk for Problem 1.

Corollary 1. The optimal portfolios in Theorem 2 are given by Lemma 5 with the expected rate of return γ_t^* in (32).

VII. NUMERICAL EXAMPLES

Let p be a positive probability. We assume R_t^i obeys a normal distribution, and then the constant κ in (24) is given by

$$\kappa = -\int_0^p \Phi^{-1}(q)\, h(q)\, dq \Big/ \int_0^p h(q)\, dq. \tag{36}$$

Here Φ is the cumulative distribution function of the standard normal distribution. We deal with the following fuzzy random variable $\tilde{R}_t^i$ in $\mathcal{A}$:

$$(\tilde{R}_t^i(\omega))_\alpha = R_t^i(\omega) + (\tilde{a}_t^i)_\alpha \tag{37}$$

for ω in Ω and α between 0 and 1, where R_t^i is a real-valued random variable with the mean $E(R_t^i)$ and $\tilde{a}_t^i$ is a fuzzy number such that $(\tilde{a}_t^i)_\alpha = [-c_t^i(1-\alpha), c_t^i(1-\alpha)]$ for α between 0 and 1 with positive constants c_t^i, which are called fuzzy factors. We investigate a case of pessimistic and necessity, i.e. λ is 1 and $w(\alpha)$ is $1-\alpha$. Let $n = 4$, i.e. 4 assets. We give the expected rates of return μ_t with fuzzy factors and a variance-covariance matrix Σ_t by Tables I and II. At time t, we can easily calculate A_t is 10.6543, B_t is 0.695848, C_t is 0.0476225 and Δ_t is 0.0231799.

Example 1. In case of $h(q) = 1$ for all q, the weighted average value-at-risk (6) is reduced to the average value-at-risk (5). We discuss a case of less than 1% part of the standard normal distribution, i.e. p is 0.01. Then κ is 2.66521 from (36). From Lemma 5 we can calculate that the optimal portfolio w_t^* for Problem 2 is (0.207187, 0.215747, 0.30793, 0.269136), and then the corresponding the expected rate of return γ_t^* is 0.0655616 and the weighted average value-at-risk $\mathrm{WAVaR}_p(E^\lambda(\tilde{R}_t))$ follows -0.751087. Namely the risk value, which we write as δ for simplicity, is 0.751087. We may also select the optimistic index, i.e. λ is 0, and the possibility evaluation weight, i.e. $w(\alpha)$ is 1. In practical financial asset management, decision maker can select optimistic index λ and possibility evaluation $w(\alpha)$ based on his preference (see Table III). From Table III we can observe that the minimum risk value δ is estimated between 0.74292 and 0.751087 and the expected rate of return γ_t^* is also estimated between 0.0655616 and 0.0737282.

Example 2. Take another weight function

$$h(q) = \frac{1}{2\sqrt{q}} \tag{38}$$

for $q \in (0,1]$. When p is 0.01, we also have κ is 2.95582 from (36), and by Lemma 5 we calculate the optimal portfolio w_t^* is (0.206852, 0.215318, 0.308563, 0.269267). The corresponding the expected rate of return is 0.065537 and the risk value δ is 0.840131.

Table I: Expected rates of return μ_t^i with fuzzy factors

	$E(R_t^i)$	c_t^i
$i = 1$	0.08	0.007
$i = 2$	0.09	0.009
$i = 3$	0.05	0.006
$i = 4$	0.07	0.007

Table II: Variance-covariance matrix with σ_t^{ij}

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	0.37	0.06	0.07	−0.06
$i = 2$	0.06	0.39	−0.08	0.09
$i = 3$	0.07	−0.08	0.35	−0.05
$i = 4$	−0.06	0.09	−0.05	0.38

Table III: The minimum risk values δ and the expected rates of return γ_t^*

	δ	γ_t^*
pessimistic and necessity	0.75109	0.0655616
optimistic and possibility	0.74292	0.0737282

Now we compare three risk criteria with value-at-risk, average value-at-risk and weighted average value-at-risk. Fig.1 illustrates the minimum risk values δ corresponding to probabilities p in cases of three cases. From Fig.1 we find the minimum risk values δ pass the risk zero line at $p \approx 0.4$ when value-at-risk and δ pass it at $p \approx 0.85$ when average value-at-risk respectively. The risk zero line implies risk-free, and these phenomena passing the risk-free line are usually abnormal. Therefore we see that the risk criterion with average value-at-risks is more reasonable than one with value-at-risks because the reasonable range for probabilities p is wide. In the same reason, the risk criterion with weighted average value-at-risks is more reasonable than one with average value-at-risks.

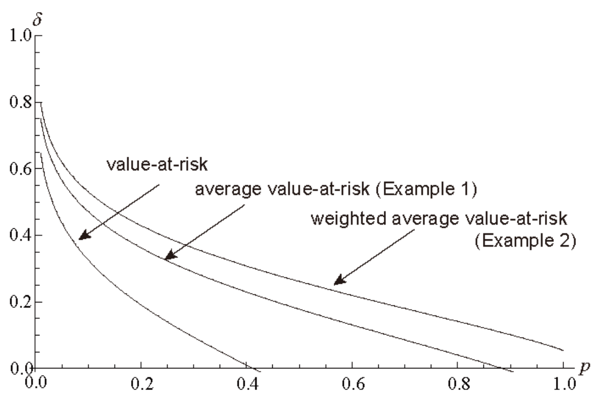

Figure 1: The minimum risk values δ

Let a discount rate $\beta = 0.93$ and let a terminal time $T = 20$ in Example 2. In dynamic case, v_t is the minimum regarding average value-at-risk between current time t and the terminal time. By Theorem 2 we observe dynamic movement of $\{ v_t \}$ in Fig.2 and we get the maximum average value-at-risks of the total asset prices v_1 is 0.139812 for Problem 1.

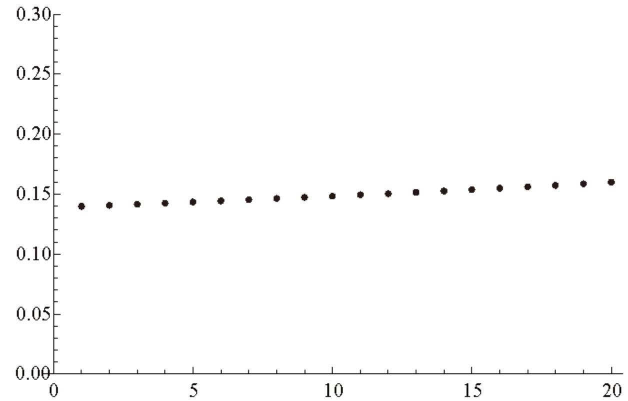

Figure 2: The sequence $\{ v_t \}$ in Theorem 2

VIII. CONCLUDING REMARKS

This paper has discussed dynamic portfolio optimization in fuzzy asset management with weighted average value-at-risks by mathematical programming and dynamic programming. The main topic is the risk-minimization of dynamic systems in the worst case of uncertainty using weighted average value-at-risks. The results in this paper will be useful for practical financial portfolios to obviate the worst sharp falling and the flash crash such as worldwide stock market panic which has heavy impact for a long time. We also found the weighted average value-at-risk gives a more reasonable criterion than average value-at-risk and value-at-risk from numerical data.

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